

Proper stacks

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Abstract

We generalize the notion of proper stack introduced by Kashiwara and Schapira to the case of a general site, and we prove that a proper stack is a stack.

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Introduction

In [3] Kashiwara and Schapira defined the notion of proper stack on a locally compact topological space X . A proper stack is a separated prestack \mathcal{S} satisfying suitable hypothesis. They proved that a proper stack is a stack. In this paper, we generalize the notion of proper stack to the case of a site X associated to a small category \mathcal{C}_X and we prove that a proper stack is a stack.

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1 Review on Grothendieck topologies and sheaves

Let \mathcal{C} be a category¹. As usual we denote by \mathcal{C}^\wedge the category of functors from \mathcal{C}^{op} to **Set** and we identify \mathcal{C} with its image in \mathcal{C}^\wedge via the Yoneda embedding. If $A \in \mathcal{C}^\wedge$, we will denote by \mathcal{C}_A the category of arrows $U \rightarrow A$ with $U \in \mathcal{C}$. When taking inductive and projective limits on a category I we will always assume that I is small.

We recall here some classical definitions (see [2]), following the presentation of [4].

Definition 1.1 *A Grothendieck topology on a small category \mathcal{C}_X is a collection of morphisms in \mathcal{C}_X^\wedge called local epimorphisms, satysfying the following conditions:*

LE1 For any $U \in \mathcal{C}_X$, $\text{id}_U : U \rightarrow U$ is a local epimorphism.

LE2 Let $A_1 \xrightarrow{u} A_2 \xrightarrow{v} A_3$ be morphisms in \mathcal{C}_X^\wedge . If u and v are local epimorphisms, then $v \circ u$ is a local epimorphism.

LE3 Let $A_1 \xrightarrow{u} A_2 \xrightarrow{v} A_3$ be morphisms in \mathcal{C}_X^\wedge . If $v \circ u$ is a local epimorphism, then v is a local epimorphism.

LE4 A morphism $u : A \rightarrow B$ in \mathcal{C}_X^\wedge is a local epimorphism if and only if for any $U \in \mathcal{C}_X$ and any morphism $U \rightarrow B$, the morphism $A \times_B U \rightarrow U$ is a local epimorphism.

Definition 1.2 *A morphism $A \rightarrow B$ in \mathcal{C}_X^\wedge is a local monomorphism if $A \rightarrow A \times_B A$ is a local epimorphism.*

A morphism $A \rightarrow B$ in \mathcal{C}_X^\wedge is a local isomorphism if it is both a local epimorphism and a local monomorphism.

Definition 1.3 *A site X is a category \mathcal{C}_X endowed with a Grothendieck topology.*

Let \mathcal{A} be a category admitting small inductive and projective limits.

Definition 1.4 *An \mathcal{A} -valued presheaf on X is a functor $\mathcal{C}_X^{op} \rightarrow \mathcal{A}$. A morphism of presheaves is a morphism of such functors. One denotes by $\text{Psh}(X, \mathcal{A})$ the category of \mathcal{A} -valued presheaves on X .*

If $F \in \text{Psh}(X, \mathcal{A})$, it extends naturally to \mathcal{C}_X^\wedge by setting

$$F(A) = \varprojlim_{(U \rightarrow A) \in \mathcal{C}_A} F(U),$$

where $A \in \mathcal{C}_X^\wedge$ and $U \in \mathcal{C}_X$.

¹We shall work in a given universe \mathcal{U} , small means \mathcal{U} -small (i.e. a set is \mathcal{U} -small if it is isomorphic to a set belonging to \mathcal{U}) and a category \mathcal{C} means a \mathcal{U} -category (i.e. $\text{Hom}_{\mathcal{C}}(X, Y)$ is \mathcal{U} -small for any $X, Y \in \mathcal{C}$).

Definition 1.5 Let X be a site.

- One says that $F \in \text{Psh}(X, \mathcal{A})$ is separated, if for any local isomorphism $A \rightarrow U$ with $U \in \mathcal{C}_X$ and $A \in \mathcal{C}_X^\wedge$, $F(U) \rightarrow F(A)$ is a monomorphism.
- One says that $F \in \text{Psh}(X, \mathcal{A})$ is a sheaf, if for any local isomorphism $A \rightarrow U$ with $U \in \mathcal{C}_X$ and $A \in \mathcal{C}_X^\wedge$, $F(U) \rightarrow F(A)$ is an isomorphism.

2 Review on stacks

Let \mathcal{C}_X be a small category. We suppose that a Grothendieck topology on \mathcal{C}_X is defined and we denote by X the associated site. We recall some classical definitions (see [1]), following the presentation of [4].

Definition 2.1 A prestack \mathcal{S} on X is the data of:

- for each $U \in \mathcal{C}_X$, a category $\mathcal{S}(U)$,
- for each $V \rightarrow U \in \mathcal{C}_U$, a functor $j_{VU*} : \mathcal{S}(U) \rightarrow \mathcal{S}(V)$,
- given $U, V, W \in \mathcal{C}_X$ and $W \rightarrow V \rightarrow U$, an isomorphism of functors $\lambda_{WVU} : j_{WV*} \circ j_{VU*} \xrightarrow{\sim} j_{WU*}$,

such that

- $j_{UU*} = \text{id}_{\mathcal{S}(U)}$,
- given $\{U_i\}_{i \in I} \in \mathcal{C}_X$, $i=1,2,3,4$ and $U_1 \rightarrow U_2 \rightarrow U_3 \rightarrow U_4$, the following diagram commutes:

$$\begin{array}{ccc} j_{12*} \circ j_{23*} \circ j_{34*} & \xrightarrow{\lambda_{234}} & j_{12*} \circ j_{24*} \\ \downarrow \lambda_{123} & & \downarrow \lambda_{124} \\ j_{13*} \circ j_{34*} & \xrightarrow{\lambda_{134}} & j_{14*} \end{array}$$

Let $\varprojlim_{U \in \mathcal{C}_X} \mathcal{S}(U)$ denote a category defined as follows. An object F of $\varprojlim_{U \in \mathcal{C}_X} \mathcal{S}(U)$ is a family $\{(F_U)_U, (\psi_u)_u\}$ where

- for any $U \in \mathcal{C}_X$, $F_U \in \text{Ob}(\mathcal{S}(U))$,
- for any morphism $U_1 \rightarrow U_2$ in \mathcal{C}_X , $\psi_{12} : j_{12*} F_{U_2} \rightarrow F_{U_1}$ is an isomorphism, such that for any sequence $U_1 \rightarrow U_2 \rightarrow U_3$ the following diagram commutes

$$\begin{array}{ccc} j_{12*} j_{23*} F_{U_3} & \xrightarrow{\psi_{23}} & j_{12*} F_{U_2} \\ \downarrow \lambda_{123} & & \downarrow \psi_{12} \\ j_{13*} F_{U_3} & \xrightarrow{\psi_{13}} & F_{U_1} \end{array}$$

Note that $\psi_{\text{id}_U} = \text{id}_{F_U}$ for any $U \in \mathcal{C}_X$.

The morphisms are defined in natural way. Let $F, G \in \varprojlim_{U \in \mathcal{C}_X} \mathcal{S}(U)$. Then

$$\text{Hom}_{\varprojlim_{U \in \mathcal{C}_X} \mathcal{S}(U)}(F, G) \simeq \varprojlim_{U \in \mathcal{C}_X} \text{Hom}_{\mathcal{S}(U)}(F_U, G_U).$$

For any $A \in \mathcal{C}_X^\wedge$, we set

$$\mathcal{S}(A) = \varprojlim_{(U \rightarrow A) \in \mathcal{C}_A} \mathcal{S}(U)$$

A morphism $\varphi : A \rightarrow B$ in \mathcal{C}_X^\wedge defines a functor $j_{AB*} : \mathcal{S}(B) \rightarrow \mathcal{S}(A)$, therefore a prestack on \mathcal{C}_X extends naturally to a prestack on \mathcal{C}_X^\wedge .

Definition 2.2 *Let X be a site.*

- A prestack \mathcal{S} on X is called *separated* if for any $U \in \mathcal{C}_X$, and for any local isomorphism $A \rightarrow U$ in \mathcal{C}_X^\wedge , $j_{AU*} : \mathcal{S}(U) \rightarrow \mathcal{S}(A)$ is fully faithful.
- A prestack \mathcal{S} on X is called a *stack* if for any $U \in \mathcal{C}_X$, and for any local isomorphism $A \rightarrow U$ in \mathcal{C}_X^\wedge , $j_{AU*} : \mathcal{S}(U) \rightarrow \mathcal{S}(A)$ is an equivalence.

Proposition 2.3 *Let \mathcal{S} be a prestack on X . Then \mathcal{S} is a stack if and only if \mathcal{S} satisfies the following conditions:*

- (i) \mathcal{S} is separated,
- (ii) for any $U \in \mathcal{C}_X$ and for any local isomorphism $A \rightarrow U$ the restriction functor $j_{AU*} : \mathcal{S}(U) \rightarrow \mathcal{S}(A)$ admits a left adjoint j_{AU}^{-1} satisfying $j_{AU*} \circ j_{AU}^{-1} \simeq \text{id}$ (or, equivalently, the functor j_{AU}^{-1} is fully faithful).

Proof. The result follows from the fact that two categories are equivalent if and only if they admit a pair of fully faithful adjoint functors. □

3 Proper stacks

Let \mathcal{C}_X be a small category. In this section we extend a result of [3] to the case of a site X associated to a small category \mathcal{C}_X .

Let \mathcal{S} be a prestack on X and assume the following hypothesis

- $$(1) \quad \begin{cases} - \text{ for any } U, V \in \mathcal{C}_X \text{ and any morphism } U \rightarrow V \text{ in } \mathcal{C}_X^\wedge, \text{ the functor } j_{UV*} : \mathcal{S}(V) \rightarrow \mathcal{S}(U) \text{ admits a left adjoint } j_{UV}^{-1} \text{ satisfying} \\ \quad \text{id}_{\mathcal{S}(U)} \xrightarrow{\sim} j_{UV*} \circ j_{UV}^{-1} \text{ (or, equivalently, } j_{UV}^{-1} \text{ is fully faithful),} \\ - \text{ for all } U \in \mathcal{C}_X \text{ the category } \mathcal{S}(U) \text{ admits small inductive limits.} \end{cases}$$

Lemma 3.1 *Let \mathcal{S} be a prestack and assume (1). Let $A \in \mathcal{C}_X^\wedge$ and $A \rightarrow V$. Then the functor j_{AV*} admits a left adjoint, denoted by j_{AV}^{-1} .*

Proof. Let $F = \{F_U\}_{(U \rightarrow A) \in \mathcal{C}_A} \in \mathcal{S}(A)$, and let $j_{AV}^{-1}F := \varinjlim_{(U \rightarrow A) \in \mathcal{C}_A} j_{UV}^{-1}F_U$.

This defines a functor $j_{AV}^{-1} : \mathcal{S}(A) \rightarrow \mathcal{S}(V)$. Let $G \in \mathcal{S}(V)$. We have the chain of isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{S}(V)}(j_{AV}^{-1}F, G) &= \mathrm{Hom}_{\mathcal{S}(V)}(\varinjlim_{(U \rightarrow A) \in \mathcal{C}_A} j_{UV}^{-1}F_U, G) \\ &\simeq \varprojlim_{(U \rightarrow A) \in \mathcal{C}_A} \mathrm{Hom}_{\mathcal{S}(V)}(j_{UV}^{-1}F_U, G) \\ &\simeq \varprojlim_{(U \rightarrow A) \in \mathcal{C}_A} \mathrm{Hom}_{\mathcal{S}(U)}(F_U, j_{UV*}G) \\ &\simeq \mathrm{Hom}_{\mathcal{S}(A)}(F, j_{AV*}G) \end{aligned}$$

□

Lemma 3.2 *Let \mathcal{S} be a prestack on X satisfying (1), let $U', U, V \in \mathcal{C}_X$ and $U' \rightarrow U \rightarrow V$. Then*

- (i) *there exists a canonical morphism $j_{U'V}^{-1} \circ j_{U'V*} \rightarrow j_{UV}^{-1} \circ j_{UV*}$,*
- (ii) *we have $j_{U'V}^{-1} \circ j_{U'V*} \simeq j_{U'V}^{-1} \circ j_{U'V*} \circ j_{UV}^{-1} \circ j_{UV*}$.*

Proof. (i) The adjunction morphism $j_{U'U}^{-1} \circ j_{U'U*} \rightarrow \mathrm{id}_{\mathcal{S}(U)}$ defines

$$j_{U'V}^{-1} \circ j_{U'V*} \simeq j_{U'V}^{-1} \circ j_{U'U}^{-1} \circ j_{U'U*} \circ j_{UV*} \rightarrow j_{UV}^{-1} \circ j_{UV*}.$$

- (ii) We have $j_{U'V*} \simeq j_{U'U*} \circ j_{UV*}$, and then

$$j_{U'V*} \circ j_{U'V}^{-1} \simeq j_{U'U*} \circ j_{UV*} \circ j_{UV}^{-1} \simeq j_{U'U*}.$$

Hence we have the chain of isomorphisms

$$j_{U'V}^{-1} \circ j_{U'V*} \circ j_{U'V}^{-1} \circ j_{UV*} \simeq j_{U'V}^{-1} \circ j_{U'U*} \circ j_{UV*} \simeq j_{UV}^{-1} \circ j_{UV*}.$$

□

Lemma 3.3 *Let \mathcal{S} be a prestack on X satisfying (1). Let $U, V, W \in \mathcal{C}_X$ and let $U \rightarrow W, V \rightarrow W$ be morphisms. Consider the diagram*

$$\begin{array}{ccc} U \times_W V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & W \end{array}$$

where $U \times_W V \in \mathcal{C}_X^\wedge$. Then there exists a canonical morphism

$$(2) \quad j_{U \times_W V}^{-1} \circ j_{U \times_W V*} \rightarrow j_{UW}^{-1} \circ j_{UW*} \circ j_{VW}^{-1} \circ j_{VW*}.$$

Proof. Since $U \times_W V \in \mathcal{C}_X^\wedge$ for each $F \in \mathcal{S}(W)$ we have

$$j_{U \times_W V} j_{W*} F = \{j_{W'W*} F\}_{(W' \rightarrow U \times_W V) \in \mathcal{C}_{U \times_W V}} \in \mathcal{S}(U \times_W V)$$

hence as in Lemma 3.1

$$j_{U \times_W V}^{-1} j_{U \times_W V} j_{W*} F \simeq \varinjlim_{(W' \rightarrow U \times_W V) \in \mathcal{C}_{U \times_W V}} j_{W'W}^{-1} j_{W'W*} F.$$

By Lemma 3.2 we have $j_{W'W}^{-1} \circ j_{W'W*} \circ j_{VW}^{-1} \circ j_{VW*} \simeq j_{W'W}^{-1} \circ j_{W'W*}$ for each $(W' \rightarrow U \times_W V) \in \mathcal{C}_{U \times_W V}$. We have natural morphisms

$$\begin{aligned} j_{U \times_W V}^{-1} j_{U \times_W V} j_{W*} &\xrightarrow{\sim} j_{U \times_W V}^{-1} j_{U \times_W V} j_{W*} \circ j_{VW}^{-1} \circ j_{VW*} \\ &\rightarrow j_{UW}^{-1} j_{UW*} \circ j_{VW}^{-1} \circ j_{VW*}. \end{aligned}$$

□

Let $U, V, W \in \mathcal{C}_X$ and let $U \rightarrow W, V \rightarrow W$ be morphisms. The morphism (2) induces a natural arrow

$$j_{U \times_W V}^{-1} j_{U \times_W V} j_{W*} \xrightarrow{\sim} j_{VW*} \circ j_{U \times_W V}^{-1} j_{U \times_W V} j_{W*} \circ j_{UW}^{-1} \rightarrow j_{VW*} \circ j_{UW}^{-1}.$$

Definition 3.4 A proper stack \mathcal{S} on X is a prestack satisfying

PRS1 \mathcal{S} is separated,

PRS2 for each $U \in \mathcal{C}_X$, $\mathcal{S}(U)$ admits small inductive limits,

PRS3 for all $U, V \in \mathcal{C}_X$ and $U \rightarrow V$ the functor $j_{UV*} : \mathcal{S}(V) \rightarrow \mathcal{S}(U)$ commutes with \varinjlim ,

PRS4 for all $U, V \in \mathcal{C}_X$ and $U \rightarrow V$ the functor $j_{UV*} : \mathcal{S}(V) \rightarrow \mathcal{S}(U)$ admits a left adjoint j_{UV}^{-1} , satisfying $\text{id}_{\mathcal{S}(U)} \xrightarrow{\sim} j_{UV*} \circ j_{UV}^{-1}$ (or, equivalently, the functor j_{UV}^{-1} is fully faithful),

PRS5 for all $V, U, W \in \mathcal{C}_X$, $U \rightarrow W$ and $V \rightarrow W$, the morphism

$$j_{U \times_W V}^{-1} j_{U \times_W V} j_{W*} \rightarrow j_{VW*} \circ j_{UW}^{-1}$$

is an isomorphism.

Remark 3.5 Here $U \times_W V \in \mathcal{C}_X^\wedge$, since we have not assumed that \mathcal{C}_X admits fiber products.

Lemma 3.6 Let us consider the following diagram

$$\begin{array}{ccc} A \times_V U & \longrightarrow & A \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

where $U, V \in \mathcal{C}_X$ and $A \in \mathcal{C}_X^\wedge$. Let \mathcal{S} be a proper stack on X . Then we have

$$j_{UV*} \circ j_{AV}^{-1} \simeq j_{A \times_V U}^{-1} j_{A \times_V U} j_{A*}.$$

Proof. Let $F = \{F_W\}_{(W \rightarrow A) \in \mathcal{C}_A} \in \mathcal{S}(A)$. We have the chain of isomorphisms

$$\begin{aligned}
j_{UV*} \circ j_{AV}^{-1} F &\simeq j_{UV*} \varinjlim_{(W \rightarrow A) \in \mathcal{C}_A} j_{WV}^{-1} F_W \\
&\simeq \varinjlim_{(W \rightarrow A) \in \mathcal{C}_A} j_{UV*} j_{WV}^{-1} F_W \\
&\simeq \varinjlim_{(W \rightarrow A) \in \mathcal{C}_A} j_{U \times_V W}^{-1} j_{U \times_V W} j_{U \times_V W} F_W \\
&\simeq \varinjlim_{(W \rightarrow A) \in \mathcal{C}_A} \varinjlim_{(W' \rightarrow W \times_V U) \in \mathcal{C}_{W \times_V U}} j_{W'U}^{-1} F_{W'} \\
&\simeq \varinjlim_{(W'' \rightarrow A \times_V U) \in \mathcal{C}_{A \times_V U}} j_{W''U}^{-1} F_{W''},
\end{aligned}$$

where the second and the third isomorphism follow from PRS3 and PRS5 respectively. The fourth isomorphism follows since $W \times_V U \in \mathcal{C}_X^\wedge$ and we have

$$j_{U \times_V W} F_W \simeq \{j_{W'W} F_W\}_{(W' \rightarrow W \times_V U) \in \mathcal{C}_{W \times_V U}} \simeq \{F_{W'}\}_{(W' \rightarrow W \times_V U) \in \mathcal{C}_{W \times_V U}}.$$

On the other hand we have $j_{A \times_V U} F \simeq \{F_{W''}\}_{(W'' \rightarrow A \times_V U) \in \mathcal{C}_{A \times_V U}}$, hence

$$j_{A \times_V U}^{-1} \circ j_{A \times_V U} F \simeq \varinjlim_{(W'' \rightarrow A \times_V U) \in \mathcal{C}_{A \times_V U}} j_{W''U}^{-1} F_{W''}.$$

□

Theorem 3.7 *Let X be a site associated to a small category \mathcal{C}_X . Let \mathcal{S} be a proper stack on X . Then \mathcal{S} is a stack.*

Proof. Let $A \rightarrow V$ be a local isomorphism. By Proposition 2.3 it is enough to show that $j_{AV*} \circ j_{AV}^{-1} \simeq \text{id}$. Let $F = \{F_{V_i}\}_{(V_i \rightarrow A) \in \mathcal{C}_A} \in \mathcal{S}(A)$. It satisfies, for each $V_i \rightarrow V_j$

$$(3) \quad j_{V_i V_j} F_{V_j} \xrightarrow{\sim} F_{V_i}.$$

We have to show that $j_{V_i V} j_{AV}^{-1} F \simeq F_{V_i}$ for each $V_i \rightarrow A$. Let us consider $V_{i_0} \rightarrow A$. By PRS5 and (3), for each $V_k \rightarrow A$ we have the chain of isomorphisms

$$j_{V_{i_0} V} j_{V_k V}^{-1} F_{V_k} \simeq j_{V_k \times_V V_{i_0} V_{i_0}}^{-1} j_{V_k \times_V V_{i_0} V_{i_0}} F_{V_k} \simeq j_{V_k \times_V V_{i_0} V_{i_0}}^{-1} j_{V_k \times_V V_{i_0} V_{i_0}} F_{V_{i_0}}.$$

Hence we obtain the isomorphism

$$j_{V_{i_0} V} j_{AV}^{-1} F \simeq j_{A \times_V V_{i_0} V_{i_0}}^{-1} j_{A \times_V V_{i_0} V_{i_0}} F_{V_{i_0}},$$

and $j_{A \times_V V_{i_0} V_{i_0}}^{-1} j_{A \times_V V_{i_0} V_{i_0}} F_{V_{i_0}} \simeq F_{V_{i_0}}$ since \mathcal{S} is separated and $A \times_V V_{i_0} \rightarrow V_{i_0}$ is a local isomorphism.

□

Example 3.8 *Let k be a field, and X a topological space (or, more generally, let X be a site associated to an ordered-set category). The prestack associating to an open set U of X the category of sheaves of k -vector spaces² on U is a proper stack.*

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²More generally, one can consider sheaves with values in a category \mathcal{A} admitting small inductive and projective limits, such that filtrant inductive limits are exact and satisfying the ICP property (see [4] for a detailed exposition).